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Numerical Procedure for the Dynamic Analysis of Truss Space Structures

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A numerical procedure, which lies between Rayleigh–Ritz and the finite element method so far applied only to aeronautical structures in both two- and three-dimensional space, is utilized for the dynamic analysis of periodic truss space structures, typical of appendices of artificial satellites. The structural components are unidimensional tube-type elements with circular cross sections. They are considered as beam elements, for which a complete model for the internal strain energy has been utilized. Three different numbers of component bays of these structures are assumed in the numerical applications, and the obtained results are analyzed. The advantages of the proposed method as the number of the component bays increases, particularly with respect to the CPU time, are pointed out comparing the obtained results with the ones of a classical finite element method numerical program, such as MSC/NASTRAN.

Nomenclature

A = beam cross-sectional area
E = modulus of elasticity
E_{f0} = parameter containing the modulus of elasticity
e_e, e_{de}, e_{ded} = occurring integrals determined by the local describing functions
G = shear rigidity modulus
G_{t0}, G_d = parameters containing the shear rigidity modulus
$g^{(n)}_{i_a i_b i_c}, g^{(r)}_{i_a i_b i_c}, g^{(s)}_{j_a j_b j_c}$ = global describing functions coefficients of the generic variable S_n
J_f = flexural moment of inertia
J_t = torsional moment of inertia
K = stiffness matrix
L = length of the generic beam
$\bar{L}$ = nondimensional length of the generic beam
L_0 = reference length
$l^{(I_b)}_{n_{ie}}, l^{(I_b)}_{r_{ie}}, l^{(I_b)}_{s_{je}}$ = local describing functions coefficients of the generic variable S_n in the I_bth beam
M = mass matrix
N = number of Lagrangian degrees of freedom
N_a, N_b, N_c = number of global describing functions along the axes X, Y, and Z, respectively
N_{el} = number of local describing functions
$P_e, P_{de}, P_{ed}, P_{ded}$ = occurring mixed integrals determined by the global and local describing functions coupling
P_p, P_{dp}, P_{dpd} = occurring integrals obtained by the global describing functions
q_i = generic Lagrangian degree of freedom
R_{ij} = rotation matrix element connecting the axis x_i with the axis X_{lj}
S_{da} = parameter containing the modulus of elasticity
S_{mt}, S_{mf}, S_{ma} = parameters containing the mass density and J_t, J_f, and A, respectively
S_n = generic independent variable
$\mathcal{T}$ = kinetic energy
U, V, W; U_1, U_2, U_3 = nondimensional displacements along the axes of the main reference system
U_l, V_l, W_l; U_{l1}, U_{l2}, U_{l3} = nondimensional displacements along the axes of the local reference system
$\mathcal{U}_a$ = axial strain energy
$\mathcal{U}_f$ = flexural strain energy
$\mathcal{U}_T$ = total strain energy
$\mathcal{U}_t$ = torsional strain energy
$\mathcal{U}_V$ = shear strain energy
u, v, w; u_1, u_2, u_3 = displacements along the axes of the main reference system
u_l, v_l, w_l; u_{l1}, u_{l2}, u_{l3} = displacements along the axes of the local reference system
X, Y, Z; X_1, X_2, X_3 = nondimensional coordinates of the main reference system
X_l, Y_l, Z_l; X_{l1}, X_{l2}, X_{l3} = nondimensional coordinates of the local reference system
x, y, z; x_1, x_2, x_3 = main reference system
x_l, y_l, z_l; x_{l1}, x_{l2}, x_{l3} = local reference system
α = azimuth angle of the oriented beam
β = elevation angle of the oriented beam
δ_{ij} = Kronecker's delta
$\theta_X, -\theta_Z, \theta_Y$; $\theta_{l1}; \theta_{l2} \cdot \theta_{l3}$ = rotations along the coordinates of the local reference system
$\theta_x, \theta_y; \theta_z$; $\theta_1, \theta_2, \theta_3$ = rotations along the coordinates of the main reference system
ρ = mass density
χ = shear factor
ω = angular frequency
ω_d = nondimensional frequency parameter

Subscripts

i_a, i_b, i_c; j_a, j_b, j_c = coefficients of the global describing functions
$i_r, j_s; i_{er}, j_{es}$ = Lagrangian degrees of freedom
n_{ie}, r_{ie}, s_{je} = coefficients of the local describing functions
X, Y, Z = rotations around the axes of the local reference system
x, y, z = rotations around the axes of the main reference system

Superscripts

(I_b) = identification number of the I_bth beam
$(n), (r), (s)$ = generic independent variable

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Introduction

THE study of the dynamic behavior of periodic structures has been developed by several authors. In most cases the proposed methods of analysis lead to generalized eigenproblem formulations that require the computation of complex eigenpairs.

Meirovitch and Remi[1] utilized a matrix difference equations model to obtain a solution to external excitations response of truss structures by the Z-transform method. Santini et al.[2] developed a numerical program, starting from the finite element method (FEM) and considering the structure as formed by a periodic series of component bay elements, connected by an interface between adjacent elements.

Renton[3] built a numerical approach to determine the elastic behavior of space trusses considering an equivalent beamlike model of the whole structure. This numerical model was performed by casting the matrix equations for pin-ended bars, which are component members with only axial rigidity, to construct the finite difference equations for the static and dynamic analysis. Sun and Liebbe[4] proposed a numerical model for the truss structures by utilizing an equivalent Timoshenko beam to simulate the whole structure.

Most of the utilized approaches consider the truss component members as pin-ended bars. An interesting and more accurate model was proposed by Anderson[5,6] and Anderson and Williams[7] for structures with repetitive geometry. In this approach the exact solution of each beam component member with transcendental describing functions was utilized to form the stiffness matrix. The method was applied to both hexahedral truss platforms and satellite antenna structures. Also, this method leads to the research of complex eigensolutions for the static and dynamic analysis of space structures. Quozzo[8] proposed an analytical method to find the exact solution with a complete equations system utilizing transcendental describing functions for each beam member component of the structure.

In this paper polynomial describing functions have been used, which give higher numerical stability. More specifically, polynomial functions are more suited for well-conditioned stiffness and mass matrices achievement.[9] A numerical procedure,[10–15] which arises from the Rayleigh–Ritz method (see Refs. 16 and 17) and which is obtained combining the Ritz analysis with the variational principles[18,19] (like FEM[9,20,21]), is applied and developed to space structures with repetitive geometry. Together with global describing functions, local describing functions of each truss member are also employed (as in FEM), which vanish at the ends of the element, but grid points are not introduced.

The stiffness and mass matrices can be formed vs the Lagrangian degrees of freedom (DOF), which are the coefficients of the global and local series describing functions. Imposing the stationary conditions of the total energy[19] leads to the generalized eigenvalue problem, which can be solved utilizing the subroutines F07FDF, F08SEF, and F02FCF of the Numerical Algorithms Group utility package.[22]

A comparison of the results obtained with the corresponding ones found by another numerical program, which utilizes the same complete model of axial, torsional, flexural, and shear beam behavior for each truss member, is used to validate the eigensolutions determined, and the FEM numerical program MSC/NASTRAN[23] is chosen.

Rotation Relations

An oriented beam in the main reference system x, y, z is shown in Fig. 1.

A local beam reference system x_l, y_l, z_l is introduced, with the axis z_l parallel to the plane x, y, which is connected with the main reference system x, y, z by the following relations:

$$x_i = x_{0i} \div R_{ij} x_{lj}, \qquad i = 1, 2, 3, \qquad j = 1, 2, 3 \tag{1a}$$

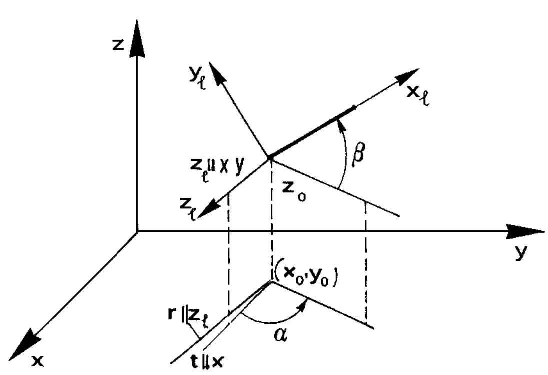


Fig. 1 Oriented beam in the main reference system *x*, *y*, *z*.

where

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_{01} = x_0, \qquad x_{02} = y_0$$

$$x_{03} = z_0, \qquad x_{l1} = x_l, \qquad x_{l2} = y_l, \qquad x_{l3} = z_l \tag{1b}$$

and the rotation matrix elements are

$$R_{11} = \cos\alpha\cos\beta, \qquad R_{12} = -\cos\alpha\sin\beta, \qquad R_{13} = \sin\alpha$$

$$R_{21} = \sin\alpha\cos\beta, \qquad R_{22} = -\sin\alpha\sin\beta, \qquad R_{23} = -\cos\alpha$$

$$R_{31} = \sin\beta, \qquad R_{32} = \cos\beta, \qquad R_{33} = 0 \tag{1c}$$

We define a nondimensional local coordinates system X_l, Y_l, Z_l:

$$X_l = x_l / L_0, \qquad Y_l = y_l / L_0, \qquad Z_l = z_l / L_0 \tag{2}$$

where L_0 is a reference length. Another nondimensional reference system X, Y, Z, which coincides with x, y, z but with a scale factor $1/L_0$, that is, $X = x/L_0$, $Y = y/L_0$, $Z = z/L_0$, is introduced. Obviously the two nondimensional reference systems X, Y, Z and X_l, Y_l, Z_l are connected by the same rotation relations. Also the length L of each beam can be reformulated in nondimensional form as $\bar{L} = L/L_0$.

It is then possible to evaluate the integral

$$\mathcal{I}(i_a i_b i_c, i_e) = \int_0^{\bar{L}} X^{i_a} Y^{i_b} Z^{i_c} X_l^{i_e - 1}\, \mathrm{d}X_l \tag{3}$$

knowledge of which is required to form the stiffness and mass matrices.

The displacements u_l, v_l, w_l, along the axes x_l, y_l, z_l, respectively, can be written in terms of the corresponding ones u, v, w along the axes x, y, z, respectively, as

$$u_{li} = R_{ji} u_j, \qquad i = 1, 2, 3, \qquad j = 1, 2, 3 \tag{4a}$$

where

$$u_{l1} = u_l, \qquad u_{l2} = v_l, \qquad u_{l3} = w_l$$

$$u_1 = u, \qquad u_2 = v, \qquad u_3 = w \tag{4b}$$

We introduce the nondimensional displacements U, V, W along x, y, z

$$U = u/L_0, \qquad V = v/L_0, \qquad W = w/L_0 \tag{5}$$

and the corresponding displacements U_l, V_l, W_l along x_l, y_l, z_l,

$$U_l = u_l / L_0, \qquad V_l = v_l / L_0, \qquad W_l = w_l / L_0 \tag{6}$$

Obviously the relations between U_l, V_l, W_l and U, V, W are the same as in Eq. (4a).

Then the rotations θ_X, θ_Y, θ_Z around the axes x_l, z_l, $-y_l$, respectively, are introduced, which are connected with the rotations θ_x, θ_y, θ_z around the axes x, y, z, respectively, by the same rotation relations as in Eq. (4a):

$$\theta_{li} = R_{ji}\theta_j, \qquad i = 1, 2, 3, \qquad j = 1, 2, 3 \tag{7a}$$

where

$$\theta_{l1} = \theta_X, \qquad \theta_{l2} = -\theta_Z, \qquad \theta_{l3} = \theta_Y$$

$$\theta_1 = \theta_x, \qquad \theta_2 = \theta_y, \qquad \theta_3 = \theta_z \tag{7b}$$

Mathematical Model

The expression of the flexural strain energy of the generic beam reads

$$\mathcal{U}_f = \frac{1}{2}\frac{EJ_f}{L_0}\int_0^{\bar{L}} \left[\left(\frac{\partial\theta_Y}{\partial X_l}\right)^2 + \left(\frac{\partial\theta_Z}{\partial X_l}\right)^2\right] \mathrm{d}X_l \tag{8}$$

The expression of the shear strain energy can be written as

$$\mathcal{U}_V = \frac{1}{2}\frac{GA}{\chi}L_0\int_0^{\bar{L}}\left[\left(\theta_Y - \frac{\partial V_l}{\partial X_l}\right)^2 + \left(\theta_Z - \frac{\partial W_l}{\partial X_l}\right)^2\right]\mathrm{d}X_l \quad (9)$$

The torsional strain energy expression is defined as follows:

$$\mathcal{U}_t = \frac{1}{2}\frac{GJ_t}{L_0}\int_0^{\bar{L}}\left(\frac{\partial\theta_X}{\partial X_l}\right)^2\mathrm{d}X_l \quad (10)$$

The axial vibration energy can be written as

$$\mathcal{U}_a = \frac{1}{2}EAL_0\int_0^{\bar{L}}\left(\frac{\partial U_l}{\partial X_l}\right)^2\mathrm{d}X_l \quad (11)$$

Then in expressions (8–11) we substitute in place of θ_X, θ_Y, θ_Z and U_l, V_l, W_l their relations vs θ_x, θ_y, θ_z and U, V, W, respectively, for which we assume a representation in form of series expansions of polynomial global and local describing functions:

$$S_n = \sum_{i_a i_b i_c} g^{(n)}_{i_a i_b i_c} X^{i_a} Y^{i_b} Z^{i_c} + \sum_{i_e=1}^{N_{\mathrm{el}}} l^{(I_b)}_{n_{ie}} X_l^{i_e}(\bar{L} - X_l)$$

$$i_a = 1, 2, \ldots, N_a, \qquad i_b = 0, 1, 2, \ldots, N_b - 1$$

$$i_c = 0, 1, 2, \ldots, N_c - 1 \quad (12a)$$

considering that the clamped edge is supposed at $X = 0$, and

$$n_{ie} = (n-1)N_{\mathrm{el}} + i_e, \qquad I_b = 1, 2, \ldots, N_{\mathrm{beams}} \quad (12b)$$

where the generic independent variable S_n corresponds to U, V, W, θ_x, θ_y, and θ_z for $n = 1, 2, \ldots, 6$, respectively, N_{el} is the number of local describing functions in every single beam, and I_b is the identification number of the generic beam, which can be at most equal to the whole number of beams N_{beams}. For this reason, we have that the whole number of Lagrangian DOF is

$$N = N^* + 6N_{\mathrm{beams}}N_{\mathrm{el}} \quad (13)$$

where $N^* = 6N_aN_bN_c$. After substitution of the series expansions (12a) into Eqs. (8–11) and taking into account the rotation relations between differential operators

$$\frac{\partial[\,]}{\partial X_l} = R_{i1}\frac{\partial[\,]}{\partial X_i}, \qquad i = 1, 2, 3 \quad (14a)$$

where

$$X_1 = X, \qquad X_2 = Y, \qquad X_3 = Z \quad (14b)$$

we obtain a series of integrals as in Eq. (3), which can be evaluated (see Appendix A), and consequently the total strain energy

$$\mathcal{U}_T = \mathcal{U}_f + \mathcal{U}_V + \mathcal{U}_t + \mathcal{U}_a \quad (15)$$

can be determined vs the Lagrangian DOF

$$\mathcal{U}_T = \frac{1}{2}\sum_{ij} k_{ij}q_iq_j \quad (16)$$

where k_{ij} are the elements of the stiffness matrix K (see Appendix B).

The generalized DOF q_i can be defined as follows:

$$q_i = g^{(n)}_{i_a i_b i_c} \quad (17a)$$

for

$$i \leq N^* \quad (17b)$$

and consequently,

$$i = (i_a - 1)N_bN_c + i_bN_c + i_c + (n-1)(N^*/6) + 1 \quad (17c)$$

whereas for

$$N^* < i \leq N \quad (17d)$$

we have

$$q_i = l^{(I_b)}_{n_{ie}} \quad (17e)$$

and also

$$i = N^* + (I_b - 1)6N_{\mathrm{el}} + n_{ie} \quad (17f)$$

Also the total kinetic energy

$$\mathcal{T} = \frac{1}{2}\rho\omega^2L_0\int_0^{\bar{L}}\left[AL_0^2U^2 + J_t\theta_X^2 + J_f\left(\theta_Y^2 + \theta_Z^2\right)\right]\mathrm{d}X_l \quad (18)$$

can be expressed vs the Lagrangian DOF as

$$\mathcal{T} = \frac{1}{2}\omega^2\sum_{ij} m_{ij}q_iq_j \quad (19)$$

where m_{ij} are the elements of the mass matrix M (see Appendix B). Last, by minimizing the total energy,[18,19] we arrive at the generalized eigenvalue problem:

$$(K - \omega^2M)Q = 0 \quad (20)$$

the solution of which is found by appropriate algorithms.

Applications

The geometric shape of the periodic structure with two component bays and the cross section of the generic beam are shown in Fig. 2.

The shell thickness can be neglected with respect to the radius of the cross section for the computation of the torsional and flexural moment of inertia.

The frequency and modal shapes computation has been performed for the three-, four-, and five-bays cases, respectively.

Results

The nondimensional frequency parameter ω_d is connected with the true frequency ω via the following relation:

$$\omega_d^2 = \omega^2\left(\rho L_0^2/E\right)$$

Table 1 refers to the case of three-component bays of the periodic structure. The values of the first three vibration modes (VM)

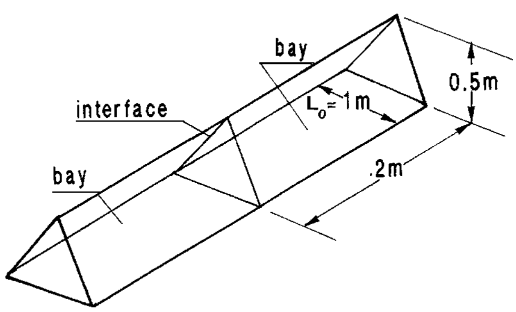
Table 1 Vibration frequencies obtained in the first case

Numerical parameter	PM	PM	PM	FEM	FEM	FEM	FEM	FEM	FEM	FEM
N	306	324	378	756	918	1242	1566	3186	6426	12906
CPU, s	3.0	3.54	5.23	7.1	8.8	11.5	12.2	24.4	48.5	103.2
ω_d										
First VM	0.001884	0.001884	0.001884	0.001765	0.001795	0.001827	0.001842	0.001867	0.001876	0.001880
Second VM	0.002125	0.002125	0.002125	0.002021	0.002047	0.002074	0.002087	0.002108	0.002116	0.002120
Third VM	0.003771	0.003771	0.003771	0.003711	0.003723	0.003736	0.003744	0.003756	0.003763	0.003766

have been obtained both by the proposed method (PM) and the FEM, utilizing a MSC/NASTRAN[23] program (version 70 and with lumped mass formulation). The unidimensional elements utilized in the MSC/NASTRAN program are BEAM, with constant cross-sectional geometric characteristics.

The FEM numerical test matrix that specifies the number of elements into which both the longitudinal (long) and transverse beams (short) are divided is given in Table 2. In Table 3, the PM numerical test matrix, which reports the number of global and local describing functions vs the number N of Lagrangian DOF is shown.

Table 4 shows the same frequency parameter values in the second case with four bays. The corresponding FEM and PM numerical test matrices are shown in Tables 5 and 6, respectively. The results obtained in the third case with five component bays are reported in Table 7, whereas Tables 8 and 9 show the corresponding FEM and PM numerical test matrices, respectively.



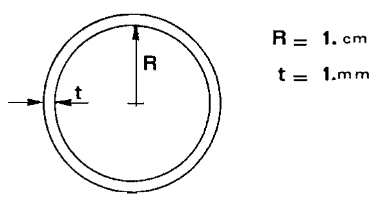


Fig. 2 Geometric shape of the periodic structure with a) two component bays and b) cross section of the generic beam member.

Table 2 FEM numerical test matrix in the first case

N	Long	Short
756	10	5
918	12	6
1,242	16	8
1,566	20	10
3,186	40	20
6,426	80	40
12,906	160	80

Table 3 PM numerical test matrix in the first case

N	N_a	N_b	N_c	N_{el}
306	5	1	3	2
324	6	1	3	2
378	3	1	3	3

It is possible to see that the results of PM converge very quickly toward their limit values, and for this reason a graphical test of the behavior of the frequency value vs $1/N$ is not necessary, which instead is very useful in the case of FEM. If we look at the Fig. 3, where the behavior of the first nondimensional frequency value vs $1/N$ in the first case considered is shown, we can notice that the value of the FEM converges toward the PM value. In Fig. 3, the FEM results of the first frequency, shown in Table 1, appear as marked dots.

The behaviors of the first frequency in the two other cases and of the second and third frequency values vs $1/N$ are very similar to that of the first frequency of the first case, and for this reason they have not been reported in the text.

The modal shapes obtained by both methods have to be considered. Such modal behaviors have been obtained by FEM by dividing the longitudinal beams into 20 elements and the transverse beams into 10 elements. In the same grid points of the FEM model, the displacements values along the three axes of the main reference system have been computed by the describing functions utilized in the PM, and the modal shapes have been built for a comparison with those of FEM. The grid points at the clamped edge, with canceled DOF, are marked in all of the modal shapes drawings.

In the text, only the modal shapes obtained by the PM have been shown because the corresponding ones obtained by FEM are identical. Moreover only the modal shapes obtained with five bays are reported because the others are very similar. In Figs. 4–6 the modal behaviors corresponding to the first, second, and third frequency, respectively, obtained by PM with $N = 612$ are shown.

Discussion

The results obtained will be discussed and the reason for which three cases with increasing number of component bays have been chosen will be explained.

In the first case, if FEM is utilized $N = 12{,}906$ DOF with 103.2-s CPU time are necessary to obtain very accurate values of the frequency, whereas $N = 306$ DOF and 3.0-s CPU time are sufficient with PM. In the second case $N = 17{,}208$ DOF and 157.0-s CPU time are necessary to obtain very accurate values of the frequency with FEM, whereas $N = 504$ and 11.0-s CPU time are sufficient if we utilize the PM. In the third case, $N = 21{,}510$ DOF and 328.7-s CPU time are required by FEM to obtain very accurate results in comparison with $N = 630$ and 19.84-s CPU time necessary with PM. From

Table 5 FEM numerical test matrix in the second case

N	Long	Short
1,008	10	5
1,224	12	6
1,656	16	8
2,088	20	10
4,248	40	20
8,568	80	40
17,208	160	80

Table 6 PM numerical test matrix in the second case

N	N_a	N_b	N_c	N_{el}
486	3	1	3	3
504	4	1	3	3
522	5	1	3	3

Table 4 Vibration frequencies obtained in the second case

Numerical parameter	PM	PM	PM	FEM	FEM	FEM	FEM	FEM	FEM	FEM
N	486	504	522	1,008	1,224	1,656	2,088	4,248	8,568	17,208
CPU, s	9.95	11.0	12.10	9.1	10.0	12.2	15.0	29.3	69.6	157.0
ω_d										
First VM	0.001385	0.001380	0.001380	0.001291	0.001316	0.001341	0.001352	0.001369	0.001375	0.001377
Second VM	0.001580	0.001577	0.001577	0.001499	0.001520	0.001541	0.001551	0.001566	0.001571	0.001574
Third VM	0.002828	0.002824	0.002824	0.002787	0.002795	0.002803	0.002808	0.002816	0.002819	0.002822

Table 7 Vibration frequencies obtained in the third case

Numerical parameter	PM	PM	PM	FEM	FEM	FEM	FEM	FEM	FEM	FEM
N	612	630	648	1,260	1,530	2,070	2,610	5,310	10,710	21,510
CPU, s	17.95	19.84	21.26	10.1	11.4	14.2	18.8	36.4	100.3	328.7
ω_d										
First VM	0.001087	0.001086	0.001086	0.001013	0.001035	0.001056	0.001065	0.001078	0.001082	0.001084
Second VM	0.001252	0.001252	0.001252	0.001188	0.001206	0.001224	0.001232	0.001244	0.001248	0.001250
Third VM	0.002258	0.002257	0.002257	0.002231	0.002237	0.002243	0.002246	0.002252	0.002254	0.002256

Table 8 FEM numerical test matrix in the third case

N	Long	Short
1,260	10	5
1,530	12	6
2,070	16	8
2,610	20	10
5,310	40	20
10,710	80	40
21,510	160	80

Table 9 PM numerical test matrix in the third case

N	N_a	N_b	N_c	N_{el}
612	4	1	3	3
630	5	1	3	3
648	6	1	3	3

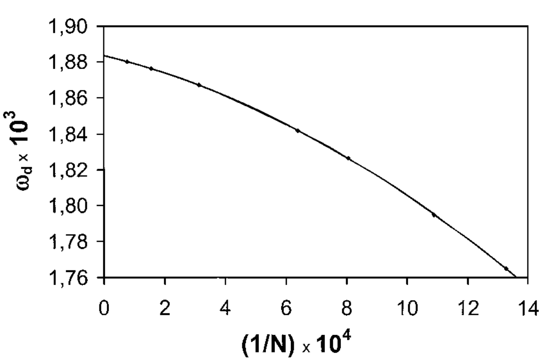


Fig. 3 Behavior of the first frequency obtained by the FEM vs 1/N in the first considered case.

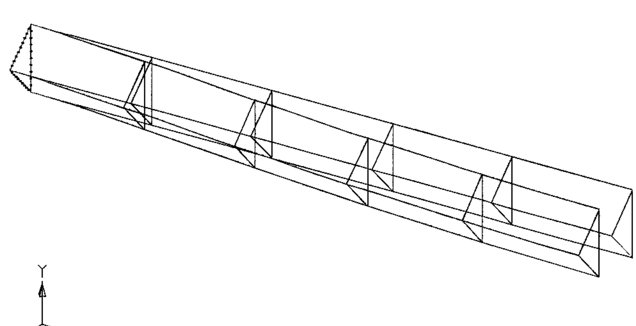


Fig. 4 First out-of-plane x, y flexural mode shape corresponding to the first frequency, obtained in the third case by the PM with N = 612.

this, we have an immediate consequence: the higher the number of component bays, the bigger the advantages of PM. This means that the convenience of utilizing this procedure, both for the CPU time and for the core storage requirements of the computer (which depend on the number N of DOF), becomes greater as the number of component bays increases. We can notice also that, if high accuracy of the obtained frequency results is not requested, in the third case similar frequency values are obtained approximately with the same CPU time by both methods (18.8 and 17.95 s, respectively), but also

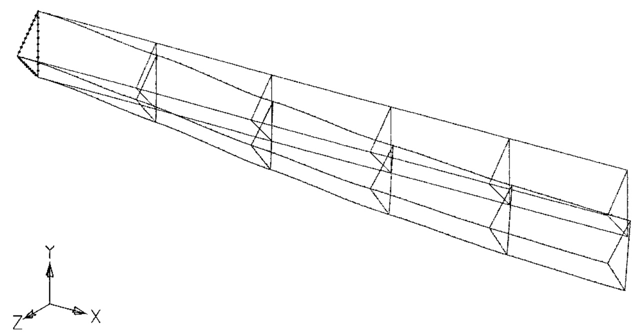


Fig. 5 First in-plane x, y flexural mode shape corresponding to the second frequency, obtained in the third case by the PM with N = 612.

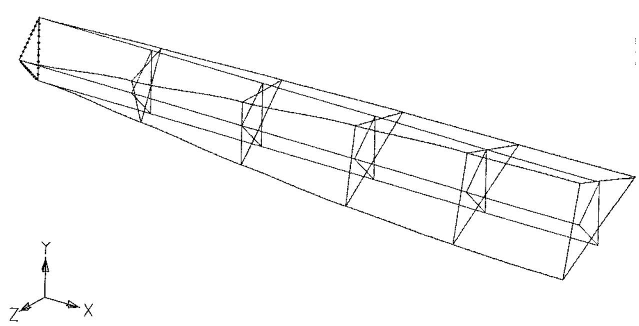


Fig. 6 First torsional mode shape corresponding to the third frequency, obtained in the third case by the PM with N = 612.

in this case a number of DOF much higher with FEM than with PM is necessary (2610 and 612 DOF, respectively).

Conclusions

The advantages just noted become more consistent if higher-order vibration modes are requested because the CPU time of the MSC/NASTRAN program increases with the number of eigensolutions required much more than the corresponding time of PM. The geometric shape of the periodic structure appears to be simple in the present paper, but the numerical program can be easily adapted to more complex shaped space truss structures to be applied for the dynamic analysis of all of the truss structural elements of satellites or space stations. In the same way the utilized numerical model can be properly modified to be applied also to structures with repetitive geometry not exactly periodic. For example, with cross-beam elements, the length varies linearly vs the structure axial coordinate, that is, instead of prismatic structures we could have pyramidal structures.

We can conclude by emphasizing that this method offers a lot of advantages in comparison with the traditional FEM numerical models, without the necessity of finding complex eigensolutions as most of the other approaches recently developed, and takes advantage of the repetitive geometry of space truss structures.

Appendix A: Evaluation of the Occurring Integrals

The integral in Eq. (3) has to be evaluated. Equation (1a) can be written in nondimensional form:

$$X_i = X_{0i} + R_{ij} X_{lj} \tag{A1}$$

The dependence of X–Y–Z on the coordinates Y_l–Z_l is not considered because we move along the axis X_l to compute the integral in Eq. (3), and we can write

$$X^{i_a} = \sum_{l_a=0}^{i_a} \binom{i_a}{l_a} (R_{11})^{l_a} X_l^{l_a} X_0^{i_a - l_a}$$

$$Y^{i_b} = \sum_{l_b=0}^{i_b} \binom{i_b}{l_b} (R_{21})^{l_b} X_l^{l_b} Y_0^{i_b - l_b}$$

$$Z^{i_c} = \sum_{l_c=0}^{i_c} \binom{i_c}{l_c} (R_{31})^{l_c} X_l^{l_c} Z_0^{i_c - l_c} \tag{A2}$$

By substituting Eq. (A2) into the expression of the integral in Eq. (3), we obtain

$$\mathcal{I}(i_a, i_b, i_c, i_e) = \sum_{l_a=0}^{i_a}\sum_{l_b=0}^{i_b}\sum_{l_c=0}^{i_c} (R_{11})^{l_a}(R_{21})^{l_b}(R_{31})^{l_c} X_0^{i_a - l_a} Y_0^{i_b - l_b} Z_0^{i_c - l_c} \times \frac{\bar{L}^{l_a + l_b + l_c + i_e}}{l_a + l_b + l_c + i_e} \tag{A3}$$

We introduce the generic polynomial

$$P(i_a, i_b, i_c) = X^{i_a} Y^{i_b} Z^{i_c} \tag{A4}$$

and

$$P_d(i_a, i_b, i_c) = \frac{\partial P(i_a, i_b, i_c)}{\partial X_l} \tag{A5}$$

which for the rotation Eq. (14a) is equal to

$$P_d(i_a, i_b, i_c) = (i_a\delta_{i1} + i_b\delta_{i2} + i_c\delta_{i3})R_{i1} P(i_a - \delta_{i1}, i_b - \delta_{i2}, i_c - \delta_{i3}) \qquad i = 1, 2, 3 \tag{A6}$$

We introduce the integral

$$P_p(i_a, i_b, i_c, j_a, j_b, j_c) = \int_0^{\bar{L}} P(i_a, i_b, i_c) P(j_a, j_b, j_c)\, \mathrm{d}X_l \tag{A7}$$

which, taking into account Eqs. (3) and (A4), is equal to

$$P_p(i_a, i_b, i_c, j_a, j_b, j_c) = \mathcal{I}(i_a + j_a, i_b + j_b, i_c + j_c, 1) \tag{A8}$$

Then we define the integral

$$P_{dp}(i_a, i_b, i_c, j_a, j_b, j_c) = \int_0^{\bar{L}} P_d(i_a, i_b, i_c) P(j_a, j_b, j_c)\, \mathrm{d}X_l \tag{A9}$$

which for Eqs. (A4), (A6), and (3) can be written as

$$P_{dp}(i_a, i_b, i_c, j_a, j_b, j_c) = (i_a\delta_{i1} + i_b\delta_{i2} + i_c\delta_{i3})R_{i1} \times \mathcal{I}(i_a + j_a - \delta_{i1}, i_b + j_b - \delta_{i2}, i_c + j_c - \delta_{i3}, 1) \qquad i = 1, 2, 3 \tag{A10}$$

and the integral

$$P_{dpd}(i_a, i_b, i_c, j_a, j_b, j_c) = \int_0^{\bar{L}} P_d(i_a, i_b, i_c) P_d(j_a, j_b, j_c)\, \mathrm{d}X_l \tag{A11}$$

which for the same mentioned relations becomes equal to

$$P_{dpd}(i_a, i_b, i_c, j_a, j_b, j_c) = (i_a\delta_{i1} + i_b\delta_{i2} + i_c\delta_{i3})(j_a\delta_{j1} + j_b\delta_{j2} + j_c\delta_{j3})R_{i1}R_{j1} \times \mathcal{I}(i_a + j_a - \delta_{i1} - \delta_{j1}, i_b + j_b - \delta_{i2} - \delta_{j2}, i_c + j_c - \delta_{i3} - \delta_{j3}, 1), \qquad i = 1, 2, 3, \qquad j = 1, 2, 3 \tag{A12}$$

We introduce the generic polynomial

$$e(i_e) = X_l^{i_e}(\bar{L} - X_l) \tag{A13}$$

and

$$e_d(i_e) = \frac{\partial e(i_e)}{\partial X_l} \tag{A14}$$

which is equal to

$$e_d(i_e) = i_e X_l^{i_e - 1}\bar{L} - (i_e + 1)X_l^{i_e} \tag{A15}$$

We introduce the integral

$$P_e(i_a, i_b, i_c, j_e) = \int_0^{\bar{L}} P(i_a, i_b, i_c) e(j_e)\, \mathrm{d}X_l \tag{A16}$$

which for relations (A4), (A13), and (3) is equal to

$$P_e(i_a, i_b, i_c, j_e) = \bar{L}\mathcal{I}(i_a, i_b, i_c, j_e + 1) - \mathcal{I}(i_a, i_b, i_c, j_e + 2) \tag{A17}$$

Then we define the integral

$$P_{de}(i_a, i_b, i_c, j_e) = \int_0^{\bar{L}} P_d(i_a, i_b, i_c) e(j_e)\, \mathrm{d}X_l \tag{A18}$$

which, taking into account Eqs. (A4), (A6), (A13), and (3), can be written as

$$P_{de}(i_a, i_b, i_c, j_e) = (i_a\delta_{i1} + i_b\delta_{i2} + i_c\delta_{i3})R_{i1} \times [\bar{L}\mathcal{I}(i_a - \delta_{i1}, i_b - \delta_{i2}, i_c - \delta_{i3}, j_e + 1) - \mathcal{I}(i_a - \delta_{i1}, i_b - \delta_{i2}, i_c - \delta_{i3}, j_e + 2)] \qquad i = 1, 2, 3 \tag{A19}$$

and

$$P_{ed}(i_a, i_b, i_c, j_e) = \int_0^{\bar{L}} P(i_a, i_b, i_c) e_d(j_e)\, \mathrm{d}X_l \tag{A20}$$

which, for Eqs. (A4), (A15), and (3) becomes

$$P_{ed}(i_a, i_b, i_c, j_e) = j_e\bar{L}\mathcal{I}(i_a, i_b, i_c, j_e) - (j_e + 1)\mathcal{I}(i_a, i_b, i_c, j_e + 1) \tag{A21}$$

We introduce the integral

$$P_{ded}(i_a, i_b, i_c, j_e) = \int_0^{\bar{L}} P_d(i_a, i_b, i_c) e_d(j_e)\, \mathrm{d}X_l \tag{A22}$$

which, taking into account Eqs. (A4), (A6), (A15), and (3) becomes equal to

$$P_{ded}(i_a, i_b, i_c, j_e) = (i_a\delta_{i1} + i_b\delta_{i2} + i_c\delta_{i3})R_{i1} \times [j_e\bar{L}\mathcal{I}(i_a - \delta_{i1}, i_b - \delta_{i2}, i_c - \delta_{i3}, j_e) - (j_e + 1)\mathcal{I}(i_a - \delta_{i1}, i_b - \delta_{i2}, i_c - \delta_{i3}, j_e + 1)] \qquad i = 1, 2, 3 \tag{A23}$$

We define the integral

$$e_e(i_e, j_e) = \int_0^{\bar{L}} e(i_e) e(j_e)\, \mathrm{d}X_l \tag{A24}$$

which, for Eq. (A13), is equal to

$$e_e(i_e, j_e) = (\bar{L})^{i_e+j_e+3}\left[\frac{1}{i_e+j_e+1} - \frac{2}{i_e+j_e+2} + \frac{1}{i_e+j_e+3}\right] \tag{A25}$$

and

$$e_{de}(i_e, j_e) = \int_0^{\bar{L}} e_d(i_e)e(j_e)\,\mathrm{d}X_l \tag{A26}$$

for which, taking into account the Eqs. (A13) and (A15), one obtains

$$e_{de}(i_e, j_e) = (\bar{L})^{i_e+j_e+2}\left[\frac{i_e}{i_e+j_e} - \frac{2i_e+1}{i_e+j_e+1} + \frac{i_e+1}{i_e+j_e+2}\right] \tag{A27}$$

Last we introduce the integral

$$e_{ded}(i_e, j_e) = \int_0^{\bar{L}} e_d(i_e)e_d(j_e)\,\mathrm{d}X_l \tag{A28}$$

which, considering Eq. (A15), can be written as

$$e_{ded}(i_e, j_e) = (\bar{L})^{i_e+j_e+1}\left[\frac{i_e j_e}{i_e+j_e-1} - \frac{i_e(j_e+1)+(i_e+1)j_e}{i_e+j_e} + \frac{(i_e+1)(j_e+1)}{i_e+j_e+1}\right] \tag{A29}$$

Appendix B: Evaluation of the Stiffness and Mass Matrices

Now the stiffness and mass matrices can be determined. All of the integrals, P_p, P_{dp}, P_{dpd}, P_e, P_{de}, P_{ed}, P_{ded}, e_e, e_{de}, and e_{ded}, evaluated in Appendix A will be utilized.

First, the contribution due only to the global describing functions will be considered. Six couples of subscripts i_r and j_s, corresponding to the coefficients of the global describing functions of the independent variables U, V, W, θ_x, θ_y, and θ_z for $r, s = 1, 2, \ldots, 6$, respectively, have to be introduced. Hence, we have

$$q_{i_r} = g^{(r)}_{i_a i_b i_c}, \qquad q_{j_s} = g^{(s)}_{j_a j_b j_c}, \qquad r, s = 1, 2, \ldots, 6 \tag{B1}$$

and, consequently,

$$i_r = (i_a - 1) \times N_b \times N_c + i_b \times N_c + i_c + (r-1) \times (N^*/6) + 1$$
$$j_s = (j_a - 1) \times N_b \times N_c + j_b \times N_c + j_c + (s-1) \times (N^*/6) + 1 \tag{B2}$$

where i_a, i_b, i_c, j_a, j_b, and j_c are the series expansions exponents of the global describing functions, introduced in the text.

If the series expansions (12a) are substituted into the flexural, shear, torsional, and axial strain energy expressions, as in Eqs. (8–11), taking into account the rotation relations, as in Eqs. (4a) and (7a), and Eqs. (15) and (16), we obtain

$$k_{i_r j_s} = R_{r-3,1}R_{s-3,1}P_{dpd}G_{t0} + (R_{r-3,3}R_{s-3,3} + R_{r-3,2}R_{s-3,2}) \times (P_{dpd}E_{f0} + G_d P_p), \qquad r, s = 4, 5, 6 \quad r \le s \tag{B3}$$

$$k_{i_r j_s} = -G_d P_{dp}[R_{r2}R_{s-3,3} - R_{r3}R_{s-3,2}] \qquad r = 1, 2, 3, \qquad s = 4, 5, 6 \tag{B4}$$

$$k_{i_r j_s} = P_{dpd}G_d[R_{r2}R_{s2} + R_{r3}R_{s3}] + P_{dpd}S_{da}R_{r1}R_{s1} \qquad r, s = 1, 2, 3, \qquad r \le s \tag{B5a}$$

where

$$G_{t0} = GJ_t/L_0, \qquad E_{f0} = EJ_f/L_0$$
$$G_d = (GA/\chi)L_0, \qquad S_{da} = EAL_0 \tag{B5b}$$

If series expansions (12a) are substituted into the expression of the kinetic energy in Eq. (18), taking into account the rotation relations and Eq. (19), we can determine the mass matrix elements

$$m_{i_r j_s} = R_{r-3,1}R_{s-3,1}P_p S_{mt} + [R_{r-3,2}R_{s-3,2} + R_{r-3,3}R_{s-3,3}]P_p S_{mf}, \qquad r, s = 4, 5, 6 \quad r \le s \tag{B6}$$

$$m_{i_r j_r} = P_p S_{ma}, \qquad r = 1, 2, 3 \tag{B7a}$$

where

$$S_{mt} = \rho J_t L_0, \qquad S_{mf} = \rho J_f L_0, \qquad S_{ma} = \rho A L_0^3 \tag{B7b}$$

There are also local describing functions, and, consequently, their contributions to the stiffness and mass matrices have to be taken into account.

First it is necessary to introduce six couples of subscripts i_{er} and j_{es} corresponding to the coefficients of the local describing functions of the independent variables U, V, W, θ_x, θ_y, and θ_z, for $r, s = 1, 2, \ldots, 6$, respectively. We have then

$$q_{i_{er}} = l^{(I_b)}_{r_{ie}}, \qquad q_{j_{es}} = l^{(I_b)}_{s_{je}}, \qquad r, s = 1, 2, \ldots, 6 \tag{B8a}$$

where

$$r_{ie} = (r-1)N_{el} + i_e, \qquad s_{je} = (s-1)N_{el} + j_e$$
$$i_e, j_e = 1, 2, \ldots, N_{el}, \qquad r, s = 1, 2, \ldots, 6 \tag{B8b}$$

and, consequently,

$$i_{er} = N^* + (I_b - 1)6N_{el} + r_{ie}$$
$$j_{es} = N^* + (I_b - 1)6N_{el} + s_{je} \tag{B9}$$

Then the mixed terms, which arise from the coupling between the global and local describing functions, have to be considered. If series expansions (12a) are substituted into the same expressions of the flexural, shear, torsional, and axial beam strain energy [Eqs. (8–11)], taking into account the rotation relations and Eqs. (15) and (16), one obtains

$$k_{i_r j_{es}} = R_{r-3,1}R_{s-3,1}P_{ded}G_{t0} + [R_{r-3,3}R_{s-3,3} + R_{r-3,2}R_{s-3,2}] \times (P_{ded}E_{f0} + G_d P_e), \qquad r, s = 4, 5, 6 \tag{B10}$$

$$k_{i_r j_{es}} = -G_d P_{de}[R_{r2}R_{s-3,3} - R_{r3}R_{s-3,2}] \qquad r = 1, 2, 3, \qquad s = 4, 5, 6 \tag{B11}$$

$$k_{i_r j_{es}} = -G_d P_{de}[R_{r-3,3}R_{s2} - R_{r-3,2}R_{s3}] \qquad r = 4, 5, 6, \qquad s = 1, 2, 3 \tag{B12}$$

$$k_{i_r j_{es}} = P_{ded}G_d[R_{r2}R_{s2} + R_{r3}R_{s3}] + P_{ded}S_{da}R_{r1}R_{s1} \qquad r, s = 1, 2, 3 \tag{B13}$$

If series expansions (12a) are substituted into the expression of the kinetic energy in Eq. (18) and taking into account the rotation relations and Eq. (19), we can determine the mass matrix elements

$$m_{i_r j_{es}} = R_{r-3,1}R_{s-3,1}P_e S_{mt} + [R_{r-3,2}R_{s-3,2} + R_{r-3,3}R_{s-3,3}]P_e S_{mf}, \qquad r, s = 4, 5, 6 \tag{B14}$$

$$m_{i_r j_{er}} = P_e S_{ma}, \qquad r = 1, 2, 3 \tag{B15}$$

Next, the contributions due only to the local describing functions have to be considered. If the series expansions (12a) are substituted into the flexural, shear, torsional, and axial strain energy expressions [Eqs. (8–11)], taking into account the rotation relations and Eqs. (15) and (16), we have

$$k_{i_{er}j_{es}} = R_{r-3,1}R_{s-3,1}e_{ded}G_{t0} + [R_{r-3,3}R_{s-3,3} + R_{r-3,2}R_{s-3,2}] \times (e_{ded}E_{f0} + G_d e_e) \qquad r, s = 4, 5, 6, \quad r \leq s \tag{B16}$$

$$k_{i_{er}j_{es}} = -G_d e_{de}[R_{r2}R_{s-3,3} - R_{r3}R_{s-3,2}] \qquad r = 1, 2, 3, \qquad s = 4, 5, 6 \tag{B17}$$

$$k_{i_{er}j_{es}} = e_{ded}G_d[R_{r2}R_{s2} + R_{r3}R_{s3}] + e_{ded}S_{da}R_{r1}R_{s1} \qquad r, s = 1, 2, 3, \qquad r \leq s \tag{B18}$$

If series expansions (12a) are substituted into the expression of the kinetic energy in Eq. (18), taking into account the rotation relations and Eq. (19), we can determine the mass matrix elements

$$m_{i_{er}j_{es}} = R_{r-3,1}R_{s-3,1}e_e S_{mt} + [R_{r-3,2}R_{s-3,2} + R_{r-3,3}R_{s-3,3}]e_e S_{mf}, \qquad r, s = 4, 5, 6, \quad r \leq s \tag{B19}$$

$$m_{i_{er}j_{er}} = e_e S_{ma}, \qquad r = 1, 2, 3 \tag{B20}$$

Last, the symmetry of both the stiffness and mass matrix has to be imposed so that also the dual elements of the ones already determined can be found:

$$k_{ji} = k_{ij} \tag{B21}$$

$$m_{ji} = m_{ij}, \qquad i = 1, N-1, \qquad j = i+1, N \tag{B22}$$

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